Indian Statistical Institute
Mid Semestral Examination- B.Math-I
Algebra-II
28th Feb. 2011

Time : 3 hours — Max. Marks : 100

Answer question **1** and any **four** from the rest.

(1) State true or false. Justify your answers.
(a) There exists a real matrix A of odd order such that $A^2 = -I$.
(b) Let A be an $n \times n$ matrix. If $A^k = 0$ for some $k \in \mathbb{N}$, then $I_n + A$ is invertible.
(c) Let A be an $m \times n$ matrix, $m < n$. The space of solutions of the linear system $AX = 0$ has dimension at least $n - m$.
(d) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear operator

$$T(x, y)^t = (y, 0)^t.$$

Then $\mathbb{R}^2 = \mathrm{Im}(T) \oplus \mathrm{Ker}(T)$.

5+5+5+5

(2) Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3)^t = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)^t.$$

(a) What is the matrix of T with respect to the standard ordered basis for $\mathbb{R}^3$?
(b) What is the matrix of T with respect to the ordered basis $\{(1, 0, 1), (-1, 2, 1), (2, 1, 1)\}$.
(c) Prove that T is invertible and give an expression for T^{-1} (as is given for T).

(3+7+10)

(3) (i) Show that there is a bijective correspondence between bases of F^n and elements of $GL_n(F)$, where F is any field. Hence compute the order of $GL_n(\mathbb{F}_p)$.
(ii) Show that index of $SL_n(\mathbb{F}_p)$ in $GL_n(\mathbb{F}_p)$ is $p - 1$.

(15+5)

(4) (a) Find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 4 & 3 & 5 & 16 \\ 6 & 6 & 13 & 13 \\ 14 & 12 & 23 & 45 \end{bmatrix}$.
(b) Find vectors X_0 and Y such that any solution of the equation

$$AX = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 3 \end{bmatrix},$$

where A is the matrix in (a), can be expressed in the form: $X_0 + \lambda Y$ where $\lambda \in \mathbb{R}$.

(6+14)

(5) (i) Let $T : V \longrightarrow V$ be a linear operator on a vector space of dimension 2. Assume that T is not multiplication by a scalar. Prove that there is a vector $v \in V$ such that $(v, T(v))$ is a basis of V, and describe the matrix of T with respect to that basis.
(ii)Show that if the sum of the entries in each row of a square matrix A is 1, then 1 is an eigen value of A.

(8+12)

(6) (a) Let W be the vector space of all $n \times n$ real matrices whose trace is zero. Find a subspace W' of $\mathbb{R}^{n\times n}$ so that $\mathbb{R}^{n\times n} = W \oplus W'$.
(b) Let V be a vector space over an infinite field F. Prove that V is not the union of finitely many proper subspaces.

(8+12)